Differentiation 8

This class deals goes through curve drawing, rate of change and to prove identities.

Curve Drawing

Find max and min points
The point of inflexion
Two other points on the curve by substitution

Example 1 Plot the curve $y = x^3 - 3x^2 + 3$

$$\frac{dy}{dx} = 3x^2 - 6x$$

For max or min $\frac{dy}{dx} = 0$ so

$$3x^2 - 6x = 0$$
$$x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

Sub $x = 0$ into $y = x^3 - 3x^2 + 3$ so $y = 3$
One turning point (0,3)

Sub $x = 2$ into $y = x^3 - 3x^2 + 3$ so $y = -1$
Other turning point (2,-1)

$$\frac{d^2y}{dx^2} = 6x - 6$$

Sub in $x = 0 \Rightarrow \frac{d^2y}{dx^2} = -6 < 0$ maximum point.

Sub in $x = 2 \Rightarrow \frac{d^2y}{dx^2} = 6 > 0$ minimum point.

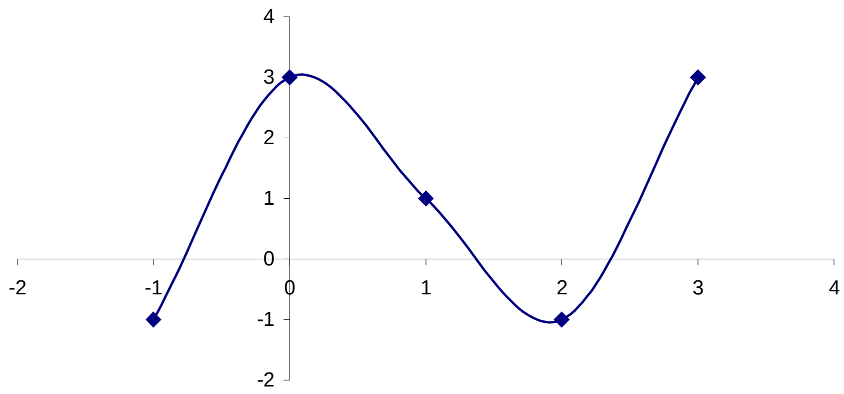
Point of inflexion

$$\frac{d^2y}{dx^2} = 6x - 6 = 0$$
$$x = 1$$

Sub $x = 1$ into $y = x^3 - 3x^2 + 3$ so $y = 1$

Note Find 2 more points, one to the left and one to the right of the max and min points.

When $x = -1$, $y = -1$ so one point is (-1,-1)
When $x = 3$, $y = 3$ so a second point is (3,3)



Asymptotes

Horizontal Asymptotes - find the limit as x tends towards infinity (divide above and below by highest power of x). This is the value of y where x does not exist.

Vertical Asymptotes - let the bottom equal zero. This is the value of x for which y does not exist.

Example 2 Draw a sketch of the curve $y = \frac{2x+4}{x-3}$ where $x \neq 3$ and $x \in R$.

Step 1 Find where it cuts x and y axes

Step 2 Differentiate to find if it is increasing or decreasing and to see if there is any stationary point.

Step 3 Find the Asymptotes

Step 1

x - axis so $y = 0$

$$\frac{2x+4}{x-3} = 0$$

$$2x + 4 = 0$$

$$x = -2 \quad \text{Point } (-2,0)$$

y - axis so $x=0$ $y=-\frac{4}{3}$ Point $\left(0,-\frac{4}{3}\right)$

Step 2 Use the quotient rule to find

$$\frac{dy}{dx}=\frac{-10}{(x-3)^2}$$

From this we find two very valuable pieces of information.

(i) The curve has no stationary points since $\frac{-10}{(x-3)^2}=0 \Rightarrow -10=0$ which is impossible.

(ii) The curve is decreasing since $\frac{dy}{dx}=\frac{-10}{(x-3)^2}<0$

Step 3

Horizontal Asymptotes

$$y=\lim_{x\to\infty}\frac{2x+4}{x-3}$$

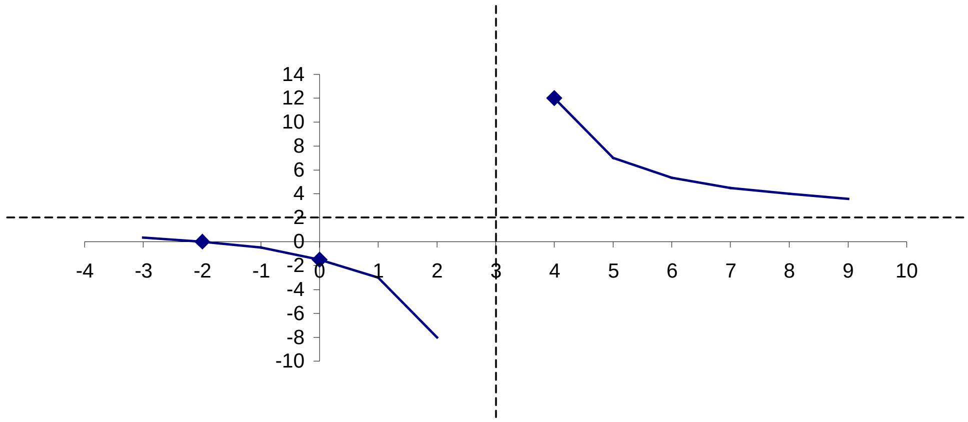
$$y=\lim_{x\to\infty}\frac{2+\frac{4}{x}}{1-\frac{3}{x}}$$

$$y=2$$

Thus $y=2$ is the horizontal Asymptotes

Vertical Asymptotes

$x-3=0 \Rightarrow x=3$ is the vertical Asymptotes



Rates of Change

Type 1 Distance, speed, acceleration.

We are given distance (s) as a function of time (t) then

speed or velocity $= \dfrac{ds}{dt}$ the rate of change of distance with respect to time.

acceleration $= \dfrac{d^2s}{dt^2}$ the rate of change of speed with respect to time.

Example 3 A particle moves along the x - axis; its distance, x meters, from the origin after t seconds is given by $x = t^3 - 3t^2 + 15t - 5$. Find the position and acceleration of the particle when it is at rest.

Note Is at rest is the same as a velocity of zero which means speed $= 0$

$$x = t^3 - 9t^2 + 15t - 5$$

$$\frac{dx}{dt} = 3t^2 - 18t + 15$$

$$3t^2 - 18t + 15 = 0$$

$$t^2 - 6t + 5 = 0$$

$$(t-1)(t-5) = 0$$

$$t = 1 \quad \text{or} \quad t = 5$$

To find the position of the particle put the values of t into $x = t^3 - 3t^2 + 12t - 5$

When $t = 1$ then $x = 5$ meters.

When $t = 5$ then $x = 105$ meters.

To find the acceleration differentiate twice and put in the values of t.

$$\frac{dx}{dt} = 3t^2 - 18t + 15$$

$$\frac{d^2x}{dt^2} = 6t - 18$$

When $t = 1$ then $\frac{d^2x}{dt^2} = -12m/s^2$

When $t = 5$ then $\frac{d^2x}{dt^2} = 12m/s^2$

Example 4 A rocket is fired up in the air. The distance s, in km, reached by the rocket after t seconds is given by $s = 12t - t^2$. Find

(i) the height of the rocket after 2 seconds,
(ii) the time taken to reach the maximum height,
(iii) the maximum height reached.

(i) To find the height of the rocket after 2 seconds sub $t = 2$ into $s = 12t - t^2$.

$$s = 12(2) - 2^2$$
$$= 20km$$

(ii) To find time taken to reach maximum height differentiate and let = 0

$$s = 12t - t^2$$
$$\frac{ds}{dt} = 12 - 2t = 0$$
$$2t = 12$$
$$t = 6$$

(iii) To find the maximum height reached sub $t = 6$ into $s = 12t - t^2$.

$$s = 12(6) - 6^2$$
$$= 36km$$

Type 2 When two rates of change are given, either directly or indirectly, and you are asked to find third rate of change related to the other two.

If for example you are given the rate of change of volume with respect to radius, $\frac{dV}{dr}$ and also given the rate of change of radius with respect to time,

$\frac{dr}{dt}$ then the rate of change of volume with respect to time is given by

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$$

m/s this is the rate of change of <u>radius</u> with respect to time $= \frac{dr}{dt}$

m^2/s this is the rate of change of <u>area</u> with respect to time $= \frac{dA}{dt}$

m^3/s this is the rate of change of <u>volume</u> with respect to time $= \frac{dV}{dt}$

Example 5 The area of a circle is increasing at a rate of $12m^2/s$ find the rate of increase of the radius when the radius is $4m$.

Told area is increasing at a rate of $12m^2/s \Rightarrow \frac{dA}{dt} = 12$

We are asked to find rate of increase of the radius with a change in time $\Rightarrow \frac{dr}{dt}$

Note Everything changes over time.

If you look in the maths table you should be able to find the formula for the area of a circle.

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

Need $\frac{dr}{dt}$ using what we have i.e. $\frac{dA}{dt}$ and $\frac{dA}{dr}$

$$\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr}$$
$$= \frac{12}{2\pi r} = \frac{6}{\pi r}$$

Sub in $r = 4$ to find the answer.

$$\frac{dr}{dt} = \frac{6}{4\pi} = \frac{3}{2\pi} m/s$$

To prove identities

Example 6 $x = a(\vartheta + \sin\vartheta)$; $y = a(1 - \cos\vartheta)$ where a is a constant.
Show

$$1 + \left(\frac{dy}{dx}\right)^2 = \sec^2\frac{\vartheta}{2}$$

Parametric differentiation

$$x = a(\vartheta + \sin\vartheta) \qquad\qquad y = a(1 - \cos\vartheta)$$

$$\frac{dx}{d\vartheta} = a(1 + \cos\vartheta) \qquad\qquad \frac{dy}{d\vartheta} = a\sin\vartheta$$

$$\frac{dy}{dx} = \frac{dy}{d\vartheta} \div \frac{dx}{d\vartheta}$$

$$= \frac{a\sin\vartheta}{a(1+\cos\vartheta)}$$

$$= \frac{\sin\vartheta}{1+\cos\vartheta}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \sec^2\frac{\vartheta}{2}$$

$$1 + \left(\frac{\sin\vartheta}{1+\cos\vartheta}\right)^2 = 1 + \tan^2\frac{\vartheta}{2}$$

We really need to show that $\dfrac{\sin\vartheta}{1+\cos\vartheta} = \tan\dfrac{\vartheta}{2}$

This is an example that has come up in several questions in the leaving cert. Because they want us to show the answer with $\tan\dfrac{\vartheta}{2}$ in it we will use the idea of $\sin\vartheta = \dfrac{2t}{1+t^2}$ and $\cos\vartheta = \dfrac{1-t^2}{1+t^2}$.

Where is this from? Well in trigonometry we use double angle formulae

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

If we half all the angles we get

$$\sin\vartheta = \frac{2\tan\frac{\vartheta}{2}}{1+\tan^2\frac{\vartheta}{2}}$$

But instead of writing this down we let $\tan\frac{\vartheta}{2} = t$, do the same for cos.

$$\frac{\sin\vartheta}{1+\cos\vartheta} = \tan\frac{\vartheta}{2}$$

$$\frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$\frac{\frac{2t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}} = \frac{2t}{2} = t = \tan\frac{\vartheta}{2}$$