# Differentiation Proofs

**Proof 1** Differentiate from first principals $y = x^2$ .

$$\begin{aligned}
f(x) &= x^2 \\
f(x+h) &= (x+h)^2 \\
f(x+h) - f(x) &= (x+h)^2 - x^2 \\
&= x^2 + 2xh + h^2 - x^2 \\
&= 2xh + h^2 \\
\frac{f(x+h) - f(x)}{h} &= 2x + h \\
\underset{h \to 0}{Lim} \frac{f(x+h) - f(x)}{h} &= 2x
\end{aligned}$$

**Proof 2** Differentiate from first principals $f(x) = x^3$

$$\begin{aligned}
f(x) &= x^3 \\
f(x+h) &= (x+h)^3 \\
f(x+h) - f(x) &= (x+h)^3 - x^3 \\
&= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\
&= 3x^2h + 3xh^2 + h^3 \\
\frac{f(x+h) - f(x)}{h} &= 3x^2 + 3xh + h^2 \\
\underset{h \to 0}{Lim} \frac{f(x+h) - f(x)}{h} &= 3x^2
\end{aligned}$$

**Proof 3** Differentiate from first principals $f(x) = \frac{1}{x}$

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$$

$$= \frac{x-(x+h)}{(x+h)(x)}$$

$$= \frac{x-x-h}{(x+h)(x)} = \frac{-h}{(x+h)(x)}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{-1}{(x+h)(x)}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{-1}{x^2}$$

**Proof 4** Differentiate from first principals $f(x)=\sqrt{x}$ .

$$f(x)=\sqrt{x}$$

$$f(x+h)=\sqrt{x+h}$$

$$f(x+h)-f(x)=\sqrt{x+h}-\sqrt{x}$$

$$=\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{h}{\sqrt{x+h}+\sqrt{x}}$$

$$\frac{f(x+h)-f(x)}{h}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

**Proof 5** Differentiate from first principals $f(x) = \sin x$

$$f(x) = \sin x$$

$$f(x+h) = \sin(x+h)$$

$$f(x+h) - f(x) = \sin(x+h) - \sin x$$

$$= 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$$

$$= 2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{h}$$

$$= 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\frac{1}{2}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 2\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\frac{1}{2}$$

$$= 2\cos\left(\frac{2x}{2}\right).1.\frac{1}{2}$$

$$= \cos x$$

**Proof 6** Differentiate from first principals $f(x)=\cos x$

$$f(x)=\cos x$$

$$f(x+h)=\cos(x+h)$$

$$f(x+h)-f(x)=\cos(x+h)-\cos x$$

$$=-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$$

$$=-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$\frac{f(x+h)-f(x)}{h}=\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$=-2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{h}$$

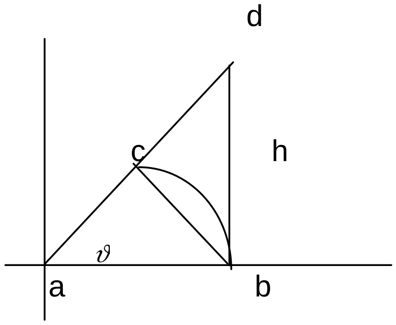
$$=-2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\frac{1}{2}$$

$$\underset{h\to 0}{Lim}\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lim}-2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\frac{1}{2}$$

$$=\text{-}2\sin\left(\frac{2x}{2}\right).1.\frac{1}{2}$$

$$=-\sin x$$

**Proof 7** To find the $\underset{\vartheta\to\infty}{Lim}\dfrac{\sin\vartheta}{\vartheta}=1$.



a is the centre of a circle so that |ab| = |ac| = r

$$\text{Area } \Delta abc \le \text{ Area sector } abc \le \text{ Area } \Delta abd$$

$$\text{Area } \Delta abc = \frac{1}{2}|ab||ac|\sin\vartheta$$

$$= \frac{1}{2}r.r\sin\vartheta = \frac{1}{2}r^2\sin\vartheta$$

$$\text{Area sector } abc = \frac{1}{2}r^2\vartheta$$

$$\text{Area } \Delta abd = \frac{1}{2}|ab|.h$$

$$\tan\vartheta = \frac{h}{|ab|}$$

$$h = |ab|\tan\vartheta \quad \text{but } |ab| = r$$

$$h = r\tan\vartheta$$

$$\text{Area } \Delta abd = \frac{1}{2}r^2\tan\vartheta$$

$$\frac{1}{2}r^2\sin\vartheta \le \frac{1}{2}r^2\vartheta \le \frac{1}{2}r^2\tan\vartheta$$

$$\sin\vartheta \le \vartheta \le \tan\vartheta$$

$$\sin\vartheta \le \vartheta \le \frac{\sin\vartheta}{\cos\vartheta} \text{ divide across by } \sin\vartheta$$

$$1 \le \frac{\vartheta}{\sin\vartheta} \le \frac{1}{\cos\vartheta}$$

$$\lim_{\vartheta\to0} 1 \le \lim_{\vartheta\to0} \frac{\vartheta}{\sin\vartheta} \le \lim_{\vartheta\to0} \frac{1}{\cos\vartheta}$$

$$1 \le \lim_{\vartheta\to0} \frac{\vartheta}{\sin\vartheta} \le 1$$

$$\Rightarrow \lim_{\vartheta\to0} \frac{\vartheta}{\sin\vartheta} = 1$$

**Proof 8** If $y = x^n$ prove $\frac{dy}{dx} = nx^{n-1}$ this is a proof by induction .

Prove for $n = 1$This must be done from first principles.

$$f(x) = x$$

$$f(x+h) = x+h$$

$$f(x+h) - f(x) = x+h-x = h$$

$$\frac{f(x+h)-f(x)}{h} = 1$$

$$\underset{h\to0}{Lim} \frac{f(x+h)-f(x)}{h} = 1$$

Assume for $n = k$

$$y = x^k \text{ then } \frac{dy}{dx} = kx^{k-1}$$

Prove for $n = k+1$

$$y = x^{k+1} \text{ then } \frac{dy}{dx} = (k+1)x^k$$

$y = x^{k+1} = x.x^k$ this must be done as a product rule.

$$\frac{dy}{dx} = 1.x^k + x.kx^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1)x^k$$

Conclusion

Since true for $n = 1$ and proven true for $n = k+1$ then true in general.

**Note** Before we go on with the next proof it is worth knowing that.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{du}{dx}=\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$\frac{dv}{dx}=\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$$

**Proof 9** To prove the sum rule

$$f(x)=u(x)+v(x)$$

$$f(x+h)=u(x+h)+v(x+h)$$

$$f(x+h)-f(x)=u(x+h)+v(x+h)-(u(x)+v(x))$$

$$=u(x+h)-u(x)+v(x+h)-v(x)$$

$$\frac{f(x+h)-f(x)}{h}=\frac{u(x+h)-u(x)+v(x+h)-v(x)}{h}$$

$$=\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}$$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{du}{dx}+\frac{dv}{dx}$$

**Proof 10** To prove the product rule.

$$f(x) = u(x)v(x)$$

$$f(x+h) = u(x+h)v(x+h)$$

$$f(x+h) - f(x) = u(x+h)v(x+h) - u(x)v(x)$$

$$= u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)$$

$$= v(x+h)\left[u(x+h) - u(x)\right] + u(x)\left[v(x+h) - v(x)\right]$$

$$\frac{f(x+h) - f(x)}{h} = \frac{v(x+h)\left[u(x+h) - u(x)\right] + u(x)\left[v(x+h) - v(x)\right]}{h}$$

$$= v(x+h)\frac{u(x+h) - u(x)}{h} + u(x)\frac{v(x+h) - v(x)}{h}$$

$$\frac{dy}{dx} = \underset{h \to 0}{Lim}\frac{f(x+h) - f(x)}{h} = v(x)\frac{du}{dx} + u(x)\frac{dv}{dx}$$

**Proof 11** To prove the quotient rule.

$$f(x)=\frac{u(x)}{v(x)}$$

$$f(x+h)=\frac{u(x+h)}{v(x+h)}$$

$$f(x+h)-f(x)=\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}$$

$$=\frac{u(x+h)v(x)-u(x)v(x+h)}{v(x+h)v(x)}$$

$$=\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{v(x+h)v(x)}$$

$$=\frac{v(x)[u(x+h)-u(x)]-u(x)[v(x+h)-v(x)]}{v(x+h)v(x)}$$

$$\frac{f(x+h)-f(x)}{h}=\frac{\dfrac{v(x)[u(x+h)-u(x)]-u(x)[v(x+h)-v(x)]}{h}}{v(x+h)v(x)}$$

$$=\frac{v(x)\dfrac{u(x+h)-u(x)}{h}-u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}$$

$$\frac{dy}{dx}=\underset{h\to 0}{Lim}\frac{f(x+h)-f(x)}{h}=\frac{v(x)\dfrac{du}{dx}-u(x)\dfrac{dv}{dx}}{v^{2}(x)}$$