1999

Question 7

Q7. (a) Find the derivative of $\sqrt{x^2+1}$

(b) (i) Let $x = t - \sin t \cos t$ and $y = 4\cos t$, $0 \le t \le \frac{\pi}{2}$.

Show that $\frac{dy}{dx} = -\frac{2}{\sin t}$.

(ii) Find the slope of the tangent to the curve

$$x^2 - y^2 - x = 1$$

at the point (2,1)

(c) Let $f(x) = x^3 + kx^2 - 4$, $x \in R$, and $k > 0$.

Show that the coordinates of the local minimum and local maximum of $f(x)$ are (0,-4) and $\left(\frac{-2k}{3}, \frac{4k^3 - 108}{27}\right)$, respectively.

Find,

(i) the range of values of k for which $f(x) = 0$ has three real roots.

(ii) the value of k for which $f(x) = 0$ has three real roots, two of which are equal.

Solution

Q7. (a) Find the derivative of $\sqrt{x^2+1}$

$$y=\sqrt{x^2+1}$$

$$y=\left(x^2+1\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)$$

$$\frac{dy}{dx}=\frac{1}{\left(x^2+1\right)^{\frac{1}{2}}}(x)$$

$$=\frac{x}{\sqrt{x^2+1}}$$

(b) (i) Let $x=t-\sin t\cos t$ and $y=4\cos t$, $0\leq t\leq\frac{\pi}{2}$.

Show that $\frac{dy}{dx}=-\frac{2}{\sin t}$.

$$x=t-\sin t\cos t$$

$$\frac{dx}{dt}=1-\left(\sin t(-\sin t)+\cos t\cos t\right)$$

$$=1-\left(-\sin^2 t+\cos^2 t\right)$$

$$=1+\sin^2 t-\cos^2 t$$

$$=2\sin^2 t$$

$$y = 4\cos t$$

$$\frac{dy}{dt} = -4\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{-4\sin t}{2\sin^2 t}$$

$$= -\frac{2}{\sin t}$$

(ii) Find the slope of the tangent to the curve

$$x^2 - y^2 - x = 1$$

at the point (2,1)

$$x^2 - y^2 - x = 1$$

$$2x - 2y\frac{dy}{dx} - 1 = 0$$

$$-2y\frac{dy}{dx} = 1 - 2x$$

$$\frac{dy}{dx} = \frac{2x-1}{2y}$$

Sub in $x = 2$ and $y = 1$

$$\frac{dy}{dx} = \frac{3}{2}$$

(c) Let $f(x) = x^3 + kx^2 - 4$, $x \in R$, and $k > 0$.

Show that the coordinates of the local minimum and local maximum of $f(x)$ are (0,-4) and $\left(\frac{-2k}{3}, \frac{4k^3 - 108}{27}\right)$, respectively.

Find,

(i) the range of values of k for which $f(x) = 0$ has three real roots

(ii) the value of k for which $f(x) = 0$ has three real roots, two of which are equal.

$f(x) = x^3 + kx^2 - 4$

$f'(x) = 3x^2 + 2kx$

$3x^2 + 2kx = 0$

$x(3x + 2k) = 0$

$x = 0$ or $x = -\frac{2k}{3}$

Sub $x = 0$ into $f(x) = x^3 + kx^2 - 4$

$f(0) = -4$ Answer is one turning point is $(0,-4)$

$f(x) = x^3 + kx^2 - 4$

$f'(x) = 3x^2 + 2kx$

$f''(x) = 6x + 2k$

Sub $x = 0$ into $f''(x) = 6x + 2k$

$f''(0) = 2k > 0$ is a minimum point since $k > 0$

Sub $x=-\frac{2k}{3}$ into $f(x)=x^3+kx^2-4$

$$f\left(-\frac{2k}{3}\right)=\left(-\frac{2k}{3}\right)^3+k\left(-\frac{2k}{3}\right)^2-4$$

$$=-\frac{8k^3}{27}+k\left(\frac{4k^2}{9}\right)-4$$

$$=-\frac{8k^3}{27}+\frac{4k^3}{9}-\frac{4}{1}$$

$$=\frac{-8k^3+12k^3-108}{27}$$

$$=\frac{4k^3-108}{27}$$

Sub $x=-\frac{2k}{3}$ into $f''(x)=6x+2k$

$$f''\left(\frac{-2k}{3}\right)=6\left(\frac{-2k}{3}\right)+2k$$

$=-4k+2k=-2k<0$ is a maximum point since $k>0$

Find,

(i) the range of values of k for which $f(x)=0$ has three real roots

Three real roots mean that the max and min are on opposite sides of the x – axis.

Minimum point is $(0,-4)$ so that means that on the maximum point that the y value must be positive.

$$\frac{4k^3 - 108}{27} > 0$$

$$4k^3 - 108 > 0$$

$$4k^3 > 108$$

$$k^3 > 27$$

$$k > 3$$

Find,

(ii) the value of k for which $f(x) = 0$ has three real roots, two of which are equal.

Three real roots, two of which are equal means the max or min are on the x – axis.

$$\frac{4k^3 - 108}{27} = 0$$

$$4k^3 - 108 = 0$$

$$4k^3 = 108$$

$$k^3 = 27$$

$$k = 3$$