1999

Question 8

Q8. (a) Find $\int\left(4x+1+\frac{1}{x^3}\right)dx$

(b) Evaluate (i) $\int_0^{\pi/6} 2\cos 4\vartheta\cos 2\vartheta d\vartheta$ (ii) $\int_{-3}^{0}(x+3)e^{x(x+6)}dx$

(c) Evaluate $\int_0^{\sqrt{3}}\sqrt{4-x^2}dx$

Hint: Let $x=2\sin\vartheta$.

Solution

Q8. (a) Find $\int\left(4x+1+\frac{1}{x^3}\right)dx$

$$\int\left(4x+1+\frac{1}{x^3}\right)dx$$

$$=\int\left(4x+1+x^{-3}\right)dx$$

$$=\frac{4x^2}{2}+x+\frac{x^{-2}}{-2}+c$$

$$=2x^2+x-\frac{1}{2x^2}+c$$

(b) Evaluate (i) $\int_0^{\pi/6} 2\cos 4\vartheta \cos 2\vartheta \, d\vartheta$ (ii) $\int_{-3}^{0} (x+3)e^{x(x+6)} dx$

(i) $\int_0^{\pi/6} 2\cos 4\vartheta \cos 2\vartheta \, d\vartheta$

$= \int_0^{\pi/6} (\cos 6\vartheta + \cos 2\vartheta) d\vartheta$

$= \left| \frac{\sin 6\vartheta}{6} + \frac{\sin 2\vartheta}{2} \right|_0^{\pi/6}$

$= \frac{\sin \pi}{6} + \frac{\sin \pi/3}{2}$

$= \frac{\frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{4}$

(ii) $\int_{-3}^{0}(x+3)e^{x(x+6)}dx$

$\int_{-3}^{0}(x+3)e^{x^2+6x}dx$ $\qquad u = x^2+6x$

$\frac{1}{2}\int_{-9}^{0}e^u du$ $\qquad du = (2x+6)dx$

$\frac{1}{2}\left|e^u\right|_{-9}^{0}$ $\qquad \frac{1}{2}du = (x+3)dx$

$\frac{1}{2}\left(e^0 - e^{-9}\right)$ $\qquad x = 0$ then $u = 0$

$\frac{1}{2}\left(1-\frac{1}{e^9}\right)$ $\qquad x = -3$ then $u = -9$

(c) Evaluate $\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$

Hint: Let $x = 2\sin\vartheta$.

$$\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$$

$$= \int_0^{\frac{\pi}{3}} 2\cos\vartheta 2\cos\vartheta\, d\vartheta$$

$$= 4\int_0^{\frac{\pi}{3}} \cos^2\vartheta\, d\vartheta$$

$$= \frac{4}{2}\int_0^{\frac{\pi}{3}} (1+\cos 2\vartheta)\, d\vartheta$$

$$= 2\left|\vartheta + \frac{1}{2}\sin 2\vartheta\right|_0^{\frac{\pi}{3}}$$

$$= 2\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} - \left(0 - \frac{1}{2}\sin 0\right)\right)$$

$$= 2\left(\frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= 2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\vartheta}$$
$$= \sqrt{4(1-\sin^2\vartheta)}$$
$$= \sqrt{4\cos^2\vartheta}$$
$$= 2\cos\vartheta$$

$$x = 2\sin\vartheta$$
$$dx = 2\cos\vartheta\, d\vartheta$$

When $x = 0$ then $2\sin\vartheta = 0$

$$\sin\vartheta = 0$$
$$\vartheta = 0$$

When $x = \sqrt{3}$ then $2\sin\vartheta = \sqrt{3}$

$$\sin\vartheta = \frac{\sqrt{3}}{2}$$
$$\vartheta = \frac{\pi}{3}$$