1996

Question 3

Q3 (a) If $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$, find a matrix M such that $M = BA^{-1}$.

(b) $P(z) = (z-2)(z^2 - 10z + 28)$

(i) Plot on an Argand diagram the solution set of $P(z) = 0$.

(ii) Verify that the three points form an equilateral triangle.

(c) (i) $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

where $i^2 = -1$.

Calculate $z_1 z_2$ in the form $x + iy$ where $x, y \in R$.

(ii) $(2+3i)(a+ib) = -1+5i$. Express $a + bi$ in the form $r(\cos\theta + i\sin\theta)$ and hence, or otherwise, calculate $(a+ib)^{11}$.

Solution

Q1 (a) If $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$, find a matrix M such that $M = BA^{-1}$.

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{9-8}\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

$$M = BA^{-1}$$

$$M = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 \\ -10 & 7 \end{pmatrix}$$

(b) $P(z) = (z-2)(z^2 - 10z + 28)$

(i) Plot on an Argand diagram the solution set of $P(z) = 0$.

(ii) Verify that the three points form an equilateral triangle.

$(z-2)(z^2 - 10z + 28) = 0$

$z - 2 = 0$

$z = 2$

$z^2 - 10z + 28 = 0$

$a = 1,\ b = -10,\ \ c = 28$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(28)}}{2(1)}$$

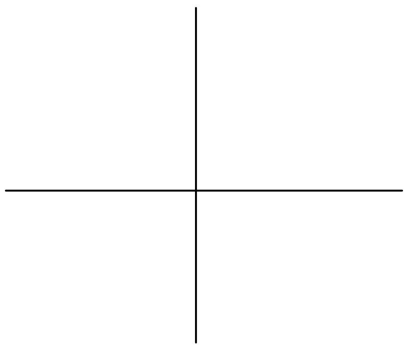
$$= \frac{10 \pm \sqrt{100 - 112}}{2}$$

$$= \frac{10 \pm \sqrt{-12}}{2}$$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$$= \frac{10 \pm 2\sqrt{3}i}{2}$$

$$= 5 \pm \sqrt{3}i$$



Points (2,0), $(5, \sqrt{3}i)$ and $(5, -\sqrt{3}i)$

Distance from $(5, \sqrt{3}i)$ and $(5, -\sqrt{3}i)$ is $2\sqrt{3}$

Distance from $(5, \sqrt{3}i)$ and (2,0) is $2\sqrt{3}$

Distance from $\left(5,-\sqrt{3}i\right)$ and (2,0) is $2\sqrt{3}$

(c) (i) $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

where $i^2 = -1$.

Calculate $z_1 z_2$ in the form $x + iy$ where $x, y \in R$.

$$z_1 z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \quad \text{since } \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

$$= 0 + 6i$$

(ii) $\left(2+3i\right)\left(a+ib\right) = -1+5i$. Express $a+bi$ in the form $r\left(\cos\theta + i\sin\theta\right)$ and hence, or otherwise, calculate $\left(a+ib\right)^{11}$.

$$\left(2+3i\right)\left(a+ib\right) = -1+5i$$

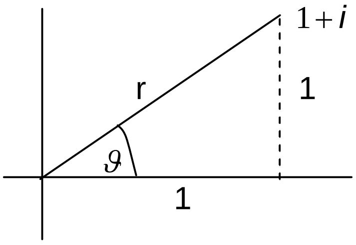
$$a + bi = \frac{-1+5i}{2+3i}$$

$$= \frac{-1+5i}{2+3i} . \frac{2-3i}{2-3i}$$

$$= \frac{-2+3i+10i-15i^2}{4-9i^2}$$

$$= \frac{13+13i}{13}$$

$$= 1 + i$$



$$r = \sqrt{1+1} = \sqrt{2}$$

$$Tan\vartheta = 1$$

$$\vartheta = \frac{\pi}{4} = 45^o$$

$$1+i = \sqrt{2}\left(\cos 45^o + i\sin 45^o\right)$$

$$(1+i)^{11} = \left(\sqrt{2}\left(\cos 45^o + i\sin 45^o\right)\right)^{11}$$

$$= \left(\sqrt{2}\right)^{11}\left(\cos 11\left(45^o\right) + i\sin 11\left(45^o\right)\right)$$

$$= \sqrt{2}\left(\sqrt{2}\right)^{10}\left(\cos 495^o + i\sin 495^o\right)$$

$$= 32\sqrt{2}\left(\cos 135^o + i\sin 135^o\right)$$

$$= 32\sqrt{2}\left(-\cos 45^o + i\sin 45^o\right)$$

$$= 32\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= 32(-1+i)$$