

Complex Numbers 4

This class deal with the very important idea of De Moivre's Theorem.

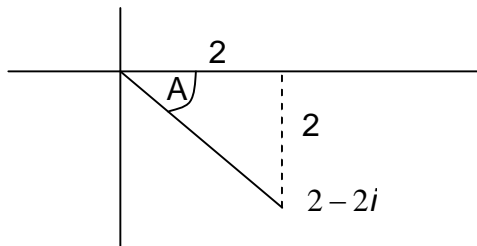
De Moivre's Theorem

$$\text{If } z = r(\cos \theta + i \sin \theta)$$

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

To raise a complex number to a power

Example 1 If $z = 2 - 2i$ express z^4 in the form $a + bi$



First write $2 - 2i$ in polar form.

$$|z| = r = \sqrt{2^2 + (-2)^2} = \sqrt{8}$$

$$\tan A = \frac{2}{2} \Rightarrow A = 45^\circ \text{ but we are in the fourth quadrant so}$$

$$\theta = 315^\circ$$

$$z = \sqrt{8}(\cos 315^\circ + i \sin 315^\circ)$$

$$z^4 = \left[\sqrt{8}(\cos 315^\circ + i \sin 315^\circ) \right]^4$$

$$= \sqrt{8}^4 (\cos 4(315^\circ) + i \sin 4(315^\circ))$$

$$= 64(\cos 1260^\circ + i \sin 1260^\circ)$$

$$= 64(\cos 180^\circ + i \sin 180^\circ)$$

$$= 64(-1 + 0i) = -64 + 0i$$

Note If we have an angle over 360° we keep on subtracting 360° until we find an angle between 0° and 360° .

To prove trigonometric equations

Example 2 Prove $\sin 4\theta = 4 \sin \theta (2 \cos^3 \theta - \cos \theta)$

Must start with $(\cos \theta + i \sin \theta)^4$ and expand in two different ways using De Moivre's Theorem and the Binomial Theorem.

Using the Binomial Theorem

$$\begin{aligned} (\cos \theta + i \sin \theta)^4 &= \cos^4 \theta + 4 \cos^3 \theta i \sin \theta + 6 \cos^2 \theta i^2 \sin^2 \theta + 4 \cos \theta i^3 \sin^3 \theta + i^4 \sin^4 \theta \\ &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \\ &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta + i(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta) \end{aligned}$$

Using De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta$$

Let the Imaginary equal the imaginary

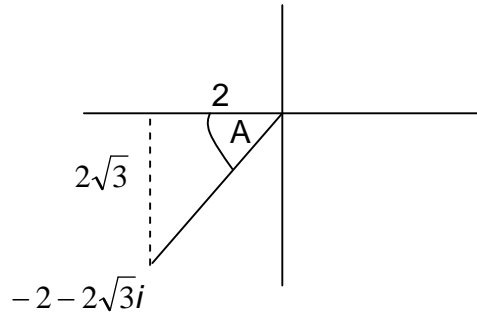
$$\begin{aligned} \sin 4\theta &= 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta && \text{take out } 4 \sin \theta \\ &= 4 \sin \theta (\cos^3 \theta - \cos \theta \sin^2 \theta) && \text{but } \sin^2 \theta = 1 - \cos^2 \theta \\ &= 4 \sin \theta (\cos^3 \theta - \cos \theta (1 - \cos^2 \theta)) \\ &= 4 \sin \theta (\cos^3 \theta - \cos \theta + \cos^3 \theta) \\ &= 4 \sin \theta (2 \cos^3 \theta - \cos \theta) \end{aligned}$$

This can be asked for sin and cos with angles 3θ , 4θ and 5θ . If asked to find an expression for $\tan 4\theta$ remember $\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$

To find the roots of a complex equation.

Example 3 If $z = (-2 - 2\sqrt{3}i)^{\frac{3}{2}}$ find two values of z in the form $a + bi$

First we must take $-2 - 2\sqrt{3}i$ and write in polar form.



$$\begin{aligned} r &= \sqrt{(-2)^2 + (-2\sqrt{3})^2} \\ &= \sqrt{4 + 12} \\ &= \sqrt{16} = 4 \end{aligned}$$

$$\begin{aligned} \tan A &= \frac{2\sqrt{3}}{2} \\ \tan A &= \sqrt{3} \\ A &= 60^\circ \\ \theta &= 240^\circ \end{aligned}$$

Because we have a fraction in the power we must use general polar form.

$$-2 - 2\sqrt{3}i = 4(\cos(240 + 2n\pi) + i \sin(240 + 2n\pi))$$

$$\begin{aligned} (-2 - 2\sqrt{3}i)^{\frac{3}{2}} &= (4(\cos(240 + 2n\pi) + i \sin(240 + 2n\pi)))^{\frac{3}{2}} \\ &= 4^{\frac{3}{2}} \left(\cos \frac{3}{2}(240 + 2n\pi) + i \sin \frac{3}{2}(240 + 2n\pi) \right) \end{aligned}$$

Since the power is $\frac{3}{2}$ we must put in just 2 consecutive values for n .

If the fraction was $\frac{1}{3}$ we would put in 3 consecutive values for n .

If the fraction was $\frac{1}{6}$ we would put in 6 consecutive values for n .

Sub in $n = 0$

$$\begin{aligned} &= 8 \left(\cos \frac{3}{2}(240) + i \sin \frac{3}{2}(240) \right) \\ &= 8(\cos 360 + i \sin 360) \\ &= 8(1 + 0i) \\ &= 8 + 0i \end{aligned}$$

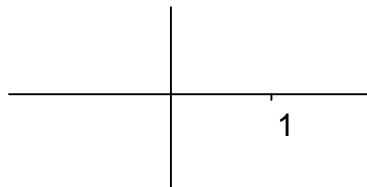
Sub in $n = 1$

$$\begin{aligned} &= 8 \left(\cos \frac{3}{2}(240 + 360) + i \sin \frac{3}{2}(240 + 360) \right) \\ &= 8 \left(\cos \frac{3}{2}(600) + i \sin \frac{3}{2}(600) \right) \\ &= 8(\cos(900) + i \sin 900) \\ &= 8(\cos 180 + i \sin 180) \\ &= 8(-1 + 0i) \\ &= -8 + 0i \end{aligned}$$

Example 4 If $z^3 = 1$ use De Moivre's Theorem to find the cube roots.

$$z^3 = 1$$

$$z = \sqrt[3]{1} = 1^{\frac{1}{3}}$$



$1 = 1 + 0i = \cos 2n\pi + i \sin 2n\pi$ since $r = 1$ and argument $\theta = 0$

$$(1 + 0i)^{\frac{1}{3}} = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{3}} \text{ using De Moivre's Theorem}$$

$$= \cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3}$$

Sub in $n = 0$ to come up with the first answer

$$\cos 0 + i \sin 0 = 1 + 0i$$

Sub in $n = 1$ to find second answer

$$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \quad \text{sub in } \pi = 180$$

$$= \cos 120 + i \sin 120 \text{ use All Silly Tom Cats to find angle in first quad}$$

$$= -\cos 60 + i \sin 60 \text{ look up maths tables page 9}$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Sub in $n = 2$ to find second answer

$$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$

$$= \cos 240 + i \sin 240$$

$$= -\cos 60 - i \sin 60$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

The three roots can be given names which can then be used in other questions.

$$1 = 1 + 0i$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Example 5 Given that 1, ω and ω^2 are the cube roots of unity

- (i) Prove that $1 + \omega + \omega^2 = 0$
 (ii) Evaluate $(1 - \omega - \omega^2)^3$

(i) $1 + \omega + \omega^2 = 0$

Sub in the values of 1, ω and ω^2 from above

$$1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i = 0$$

(ii) $(1 - \omega - \omega^2)^3$

$$\left(1 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \right)^3$$

$$\left(1 + \frac{1}{2} - \frac{\sqrt{3}}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^3$$

$$(2)^3 = 8$$

Note They have used the letter w in leaving cert questions, but this may not have anything to do with the cube roots of unity. Read the question carefully.

Trigonometry Identity

Example 6 Prove $\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3\cos \theta)$

Start with $z = \cos \theta + i \sin \theta$ and $\frac{1}{z} = \cos \theta - i \sin \theta$

$$z + \frac{1}{z} = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta = 2 \cos \theta$$

$$z - \frac{1}{z} = \cos \theta + i \sin \theta - (\cos \theta - i \sin \theta) = 2i \sin \theta$$

Now we take $2 \cos \theta = z + \frac{1}{z}$ and cube both sides since we need $\cos^3 \theta$

$$(2 \cos \theta)^3 = \left(z + \frac{1}{z}\right)^3 \quad \text{cube out right using the Binomial Theorem}$$

$$8 \cos^3 \theta = z^3 + 3z^2 \frac{1}{z} + 3z \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3$$

$$8 \cos^3 \theta = z^3 + 3z + 3 \frac{1}{z} + \frac{1}{z^3} \quad \text{bring like terms together}$$

$$8 \cos^3 \theta = z^3 + \frac{1}{z^3} + 3 \left(z + \frac{1}{z}\right)$$

$$z^3 = (\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta \quad \text{by De Moivre's Theorem}$$

$$\frac{1}{z^3} = (\cos \theta - i \sin \theta)^3 = \cos 3\theta - i \sin 3\theta \quad \text{by De Moivre's Theorem}$$

$$z^3 + \frac{1}{z^3} = \cos 3\theta + i \sin 3\theta + \cos 3\theta - i \sin 3\theta = 2 \cos 3\theta$$

Put back into the last line above

$$8 \cos^3 \theta = 2 \cos 3\theta + 3(2 \cos \theta) \quad \text{divide across by 2}$$

$$4 \cos^3 \theta = \cos 3\theta + 3 \cos \theta \quad \text{divide across by 4}$$

$$\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3 \cos \theta)$$

This can be asked for sin and cos to the powers of 3, 4 and 5.