

Complex Numbers 1

This is question 3 on paper 1. It is a very popular question for students. Like all questions some (c) parts can catch you out. Matrices goes along with this chapter.

In this class we will deal with the definition of a complex number, how to add subtract, multiply, divide complex numbers, equality and the argand diagram.

Definition of a complex number

The roots of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

but if $b^2 - 4ac < 0$ it means we have two complex roots as we have the square root of a negative number, which cannot happen.

We say any complex number consists of two parts, a real part and an imaginary part.

Example 1 Solve the equation $z^2 - 4z + 5 = 0$.

This is a quadratic equation where we need the quadratic formula to solve as from algebra.

$a = 1$, $b = -4$, $c = 5$ write down the values of a , b and c

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{-1}}{2} \quad \sqrt{-1} = i$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

The real part of $2 + i$ is 2 and the imaginary part is i .

The real part of $2 - i$ is 2 and the imaginary part is $-i$.

Notation

A complex number is usually denoted by z and is said to be of the form $z = a + bi$.

z has two parts the real part called $\text{Re}(z)$ which is a and an imaginary part $\text{Im}(z)$ which is given by bi .

$$i = \sqrt{-1}$$

$$i^2 = -1$$

Example 2 Given that $i^2 = -1$, find the value of

- (a) i^9
 (b) i^{10}

$$\begin{aligned} \text{(a)} \quad i^9 &= i^2 i^2 i^2 i \\ &= (-1)(-1)(-1)(-1)i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad i^{10} &= i^2 i^2 i^2 i^2 \\ &= (-1)(-1)(-1)(-1)(-1) \\ &= -1 \end{aligned}$$

To Add or Subtract Complex Numbers

Add real to real and imaginary to imaginary

Example 3 If $z_1 = 2 + 4i$ and $z_2 = 3 - 5i$ find the values of

(i) $z_1 + z_2$

(ii) $2z_1 - 3z_2$

$$\begin{aligned} \text{(i)} \quad z_1 + z_2 &= 2 + 4i + 3 - 5i \\ &= 5 - i \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad 2z_1 - 3z_2 &= 2(2 + 4i) - 3(3 - 5i) \\ &= 4 + 8i - 9 + 15i \\ &= -5 + 23i \end{aligned}$$

Multiplication of Complex Numbers

Same method as in algebra . One very important point is that $i^2 = -1$.

Example 4 If $z = 3 - i$ and $w = 5 + 2i$ simplify each of the following

(i) iz

(ii) zw

$$\begin{aligned} \text{(i)} \quad iz &= i(3 - i) \\ &= 3i - i^2 \\ &= 3i - (-1) \\ &= 3i + 1 \\ &= 1 + 3i \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad zw &= (3 - i)(5 + 2i) \\ &= 3(5 + 2i) - i(5 + 2i) \\ &= 15 + 6i - 5i - 2i^2 \\ &= 15 + i - 2(-1) \\ &= 15 + i + 2 \\ &= 17 + i \end{aligned}$$

Complex Conjugate

If $z = a + bi$ then the conjugate, written $\bar{z} = a - bi$. That is change the sign of the imaginary part.

Example 5 If $z = 3 + 4i$ find the value of $z\bar{z}$.

$$z = 3 + 4i \text{ then } \bar{z} = 3 - 4i$$

$z\bar{z}$ means multiply $z = 3 + 4i$ by it's conjugate $\bar{z} = 3 - 4i$.

$$\begin{aligned} z\bar{z} &= (3 + 4i)(3 - 4i) \\ &= 9 - 16i^2 \\ &= 9 - 16(-1) \\ &= 9 + 16 = 25 \end{aligned}$$

Note $\overline{z\bar{z}} = k$, where $k \in R$

Division by a Complex Number

Multiply above and below by the complex conjugate of the bottom

Example 6 Write $\frac{z_1}{z_2}$ in the form $a + bi$ where $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$

$$\frac{z_1}{z_2} = \frac{4 + 3i}{2 - 5i} \text{ multiply above and below by } 2 + 5i.$$

$$\frac{z_1}{z_2} = \frac{4 + 3i}{2 - 5i} \cdot \frac{2 + 5i}{2 + 5i} \text{ separate the question.}$$

Multiply the top by top and then the bottom by bottom and put the two answer back over each other.

Top by top

$$\begin{aligned} &(4 + 3i)(2 + 5i) && \text{split the bracket} \\ &= 8 + 20i + 6i + 15i^2 && \text{remember } i^2 = -1 \\ &= 8 + 26i - 15 && \text{add real to real and imaginary to imaginary} \\ &= -7 + 26i \end{aligned}$$

Bottom by bottom

$$(2 - 5i)(2 + 5i)$$

$$= 4 - 25i^2$$

$$= 4 + 25 = 29$$

$$\text{Answer is } \frac{-7 + 26i}{29}$$

Equality of Complex Numbers

If two complex numbers are equal then real equals real and imaginary equals imaginary.

Example 7 If $p(2 + 3i) + q(-3 + i) = -4 + 5i$ find the value of p and the value of q where $p \in R$ and $q \in R$

Multiply out and then let the real equal the real and the imaginary equal the imaginary.

$$p(2 + 3i) + q(-3 + i) = -4 + 5i$$

$$2p + 3pi - 3q + qi = -4 + 5i \quad \text{multiply out}$$

$$2p - 3q = -4 \quad \text{real equal the real}$$

$$3p + q = 5 \quad \text{imaginary equals the imaginary}$$

Solve the simultaneous equations to get $p = 1$ and $q = 2$

Example 8 Let $z = 1 + i$ and let \bar{z} be the conjugate of z .

Express $\frac{\bar{z}}{z}$ in the form $x + yi$, $x, y \in R$.

Hence solve

$$k\left(\frac{\bar{z}}{z}\right) + tz = -3 - 4i \quad \text{for real } k \text{ and real } t.$$

$$\bar{z} = 1 - i$$

$\bar{z} = 1 - i$ that is change the sign of the imaginary part.

$\frac{\bar{z}}{z} = \frac{1 - i}{1 + i}$ multiply above and below by $1 - i$ the conjugate of the bottom.

$$\frac{1 - i}{1 - i} \cdot \frac{1 - i}{1 + i} = \frac{2i}{2} = i$$

$$k\left(\frac{z}{z}\right) + tz = -3 - 4i$$

$$ki + t(1 + i) = -3 - 4i$$

$$ki + t + ti = -3 - 4i$$

Real equals the real

$$t = -3$$

Imaginary equal the imaginary

$$k + t = -4$$

$$k - 3 = -4$$

$$k = -1$$

Argand Diagram

Same as x and y axes in coordinate geometry.

The x - axis is now the real axis.

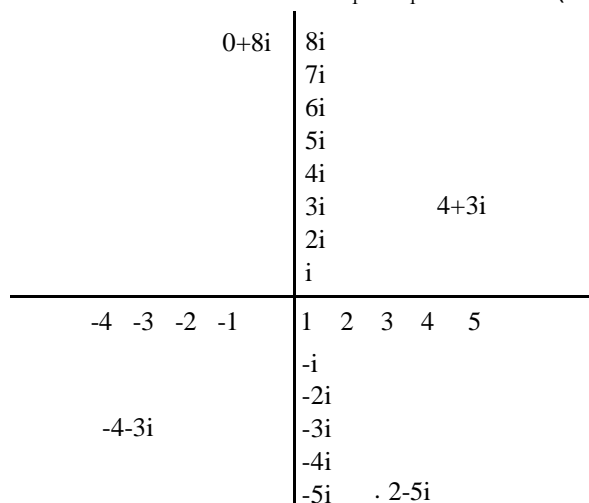
The y - axis is now the imaginary axis.

Example 9 Plot on the argand diagram z_1 , \bar{z}_2 and $z_1 - \bar{z}_1$. Find the image of z_1 under central symmetry in the origin where $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$

$$z_1 = 4 + 3i$$

$$\bar{z}_2 = 2 - 5i$$

$$z_1 - \bar{z}_1 = 3 + 4i - (3 - 4i) = 0 + 8i$$



The image of z_1 under central symmetry in the origin is $-4 - 3i$

Note All the material from the line can now be used here.