Complex Numbers Proofs

Proof 1 If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ to find an expression for

(i) $z_1 z_2$

(ii) $\dfrac{z_1}{z_2}$

(iii) $\dfrac{1}{z_1}$

$$z_1 z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$$

$$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))$$
$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$\frac{1}{z_1} = \frac{1}{r_1(\cos\theta_1 + i\sin\theta_1)}$$

$$= \frac{1}{r_1}\frac{1}{\cos\theta_1 + i\sin\theta_1}\cdot\frac{\cos\theta_1 - i\sin\theta_1}{\cos\theta_1 - i\sin\theta_1}$$

$$= \frac{1}{r_1}\frac{\cos\theta_1 - i\sin\theta_1}{\cos^2\theta_1 - i^2\sin^2\theta_1}$$

$$= \frac{1}{r_1}\frac{\cos\theta_1 - i\sin\theta_1}{\cos^2\theta_1 + \sin^2\theta_1}$$

$$= \frac{1}{r_1}(\cos\theta_1 - i\sin\theta_1)$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}.\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1\cos\vartheta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \vartheta_2) + i\sin(\theta_1 - \vartheta_2))$$

Proof 2 Proof of De Moivre's Theorem.

If $z = r(\cos\vartheta + i\sin\vartheta)$ prove $z^n = r^n(\cos n\vartheta + i\sin n\vartheta)$

We need to prove this using proof by induction.

Prove for $n = 1$

$$r(\cos\vartheta + i\sin\vartheta) = r(\cos\vartheta + i\sin\vartheta)$$

Assume for $n = k$

$$(r(\cos\vartheta + i\sin\vartheta))^k = r^k(\cos k\vartheta + i\sin k\vartheta)$$

Prove for $n = k + 1$

To prove $(r(\cos\vartheta + i\sin\vartheta))^{k+1} = r^{k+1}(\cos(k+1)\vartheta + i\sin(k+1)\vartheta)$

$$\begin{aligned}
&(r(\cos\vartheta + i\sin\vartheta))^{k+1} \\
&= (r(\cos\vartheta + i\sin\vartheta))^k (r(\cos\vartheta + i\sin\vartheta)) \\
&= r^k(\cos k\vartheta + i\sin k\vartheta)(r(\cos\vartheta + i\sin\vartheta)) \\
&= r^k.r(\cos k\vartheta\cos\vartheta + i\cos k\vartheta\sin\vartheta + i\sin k\vartheta\cos\vartheta + i^2\sin k\vartheta\sin\vartheta) \\
&= r^{k+1}(\cos k\vartheta\cos\vartheta - \sin k\vartheta\sin\vartheta + i(\cos k\vartheta\sin\vartheta + \sin k\vartheta\cos\vartheta)) \\
&= r^{k+1}(\cos(k\vartheta + \vartheta) + i\sin(k\vartheta + \vartheta)) \\
&= r^{k+1}(\cos(k+1)\vartheta + i\sin(k+1)\vartheta)
\end{aligned}$$

Conclusion

Since true for $n = 1$ and proven true for $n = k + 1$ then true in general.