

1999

Question 3

Q3 (a) If $A = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$, find A^{-1} .

(b) (i) Find a quadratic equation whose roots are $3+i$ and $3-i$, where $i^2 = -1$.

(ii) Let $P(z) = z^3 - kz^2 + 22z - 20$, $k \in R$.

$3+i$ is one root of $P(z) = 0$.

Find the value of k

Find the other two roots of the equation $P(z) = 0$.

(c) (i) Solve for w

$$\sqrt{5}|w| + iw = 3 + i.$$

Write your answer in the form $u + iv$ where $u, v \in R$.

(ii) Use De Moivre's theorem to find three roots of the equation

$$z^6 - 1 = 0$$

Solution

Q3 (a) If $A = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$, find A^{-1} .

$$A^{-1} = \frac{1}{8-5} \begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}$$

(b) (i) Find a quadratic equation whose roots are $3+i$ and $3-i$, where $i^2 = -1$.

(ii) Let $P(z) = z^3 - kz^2 + 22z - 20$, $k \in R$.

$3+i$ is one root of $P(z) = 0$.

Find the value of k

Find the other two roots of the equation $P(z) = 0$.

(i) One root is $3+i$ and a second one is $3-i$ so we can form these into a quadratic.

$$\text{Use } z^2 - (\alpha + \beta)z + \alpha\beta = 0$$

Let $\alpha = 3+i$ and $\beta = 3-i$

$$\text{Sum of roots } \alpha + \beta = 6$$

$$\text{Product of roots } \alpha\beta = (3+i)(3-i) = 9 - i^2 = 10$$

$$\text{Quadratic is } z^2 - 6z + 10 = 0$$

(ii) If $3 + i$ is a root then sub in $3 + i$ for z

$$z^3 - kz^2 + 22z - 20 = 0$$

$$(3 + i)^3 - k(3 + i)^2 + 22(3 + i) - 20 = 0$$

Must use the Binomial Theorem to expand out $(3 + i)^3$

$$\begin{aligned}(3 + i)^3 &= 3^3 + 3(3^2)i + 3(3)(i^2) + i^3 \\ &= 27 + 27i - 9 - i \\ &= 18 + 26i\end{aligned}$$

$$\begin{aligned}(3 + i)^2 &= 9 + 6i + i^2 \\ &= 8 + 6i\end{aligned}$$

$$18 + 26i - k(8 + 6i) + 22(3 + i) - 20 = 0$$

$$18 + 26i - 8k - 6ki + 66 + 22i - 20 = 0$$

$$64 + 48i - 8k - 6ki = 0 + 0i$$

Let real equal real or the imaginary equal the imaginary.

$$\begin{aligned}64 - 8k &= 0 \\ -8k &= -64 \\ k &= 8\end{aligned}$$

One root is $3 + i$ and a second one is $3 - i$ so we can form these into a quadratic. We can then divide the quadratic into the cubic to find the third root.

$$z^2 - 6z + 10 \overline{) z^3 - 8z^2 + 22z - 20}$$

The third root is $z = 2$

(c) (i) Solve for w

$$\sqrt{5}|w| + iw = 3 + i.$$

Write your answer in the form $u + iv$ where $u, v \in \mathbb{R}$.

Let $w = u + iv$

$$|w| = \sqrt{u^2 + v^2}$$

$$\sqrt{5}|w| + iw = 3 + i$$

$$\sqrt{5}\sqrt{u^2 + v^2} + i(u + iv) = 3 + i$$

$$\sqrt{5}\sqrt{u^2 + v^2} + iu - v = 3 + i$$

Let imaginary = imaginary

$$u = 1$$

Let real = real

$$\sqrt{5}\sqrt{u^2 + v^2} - v = 3$$

$$\sqrt{5}\sqrt{1 + v^2} = v + 3$$

$$5(1 + v^2) = v^2 + 6v + 9$$

$$4v^2 - 6v - 4 = 0$$

$$2v^2 - 3v - 2 = 0$$

$$(2v + 1)(v - 2) = 0$$

$$v = -\frac{1}{2} \text{ or } v = 2$$

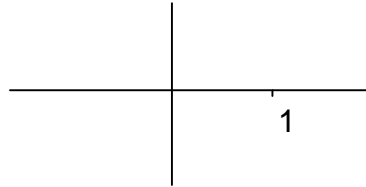
Answers are $1 + 2i$ and $1 - \frac{1}{2}i$

(ii) Use De Moivre's theorem to find three roots of the equation

$$z^6 - 1 = 0$$

$$z^6 = 1$$

$$z = \sqrt[6]{1} = 1^{\frac{1}{6}}$$



$$1 = 1 + 0i = \cos 2n\pi + i \sin 2n\pi \quad \text{since } r = 1 \text{ and argument } \vartheta = 0$$

$$(1 + 0i)^{\frac{1}{6}} = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{6}} \quad \text{using De Moivre's Theorem}$$

$$= \cos \frac{2n\pi}{6} + i \sin \frac{2n\pi}{6}$$

Sub in $n = 0$ to come up with the first answer

$$\cos 0 + i \sin 0 = 1 + 0i$$

Sub in $n = 1$ to find second answer

$$\cos \frac{2\pi}{6} + i \sin \frac{2\pi}{6}$$

$$= \cos 60 + i \sin 60$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Sub in $n = 2$ to find second answer

$$\cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6}$$

$$= \cos 120 + i \sin 120$$

$$= -\cos 60 + i \sin 60$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$