1998

Question 3

Q3 (a) Express $\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$ where $i^2 = -1$.

(b) If $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$, find AB.

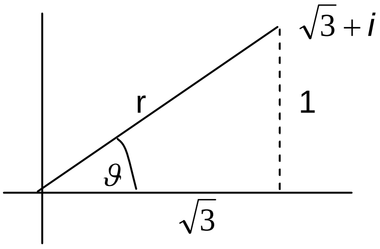
Show that $B^{-1}AB$ is of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where $p, q \in N_0$.

(c) Let $z = \cos\theta + i\sin\theta$.

Express $\dfrac{2}{1+z}$ in the form $1 - i\tan(k\theta)$, $k \in Q$ and $z \neq -1$

Solution

Q3 (a) Express $\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$ where $i^2 = -1$.



$$r = \sqrt{3+1} = \sqrt{4} = 2$$

$$Tan\vartheta = \frac{1}{\sqrt{3}}$$

$$\vartheta = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$\vartheta = \frac{\pi}{6} = 30^o$$

$$\sqrt{3+i} = 2\left(\cos 30^o + i\sin 30^o\right)$$

(b) If $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$, find AB.

Show that $B^{-1}AB$ is of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where $p, q \in N_0$.

$$AB = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 1 \\ 6 & -1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-4-1}\begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$$

$$B^{-1}AB = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 24 & 1 \\ 6 & -1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 30 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) Let $z = \cos\theta + i\sin\theta$.

Express $\dfrac{2}{1+z}$ in the form $1 - i\tan(k\theta)$, $k \in Q$ and $z \neq -1$

$$\frac{2}{1+z}$$

$$\frac{2}{1+\cos\theta + i\sin\theta} . \frac{1+\cos\theta - i\sin\theta}{1+\cos\theta - i\sin\theta}$$

$$\frac{2(1+\cos\theta - i\sin\theta)}{(1+\cos\theta)^2 - (i\sin\theta)^2}$$

$$\frac{2(1+\cos\theta - i\sin\theta)}{1+2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$\frac{2(1+\cos\theta - i\sin\theta)}{2+2\cos\theta}$$

$$\frac{1+\cos\theta - i\sin\theta}{1+\cos\theta}$$

$$= 1 - i\frac{\sin\theta}{1+\cos\theta}$$

We need to write $1 - i\dfrac{\sin\theta}{1+\cos\theta}$ in form $1 - i\tan(k\theta)$

$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ but if we let $\dfrac{\theta}{2} = t$ then it is a lot easier

$\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$.

$$\frac{\sin\theta}{1+\cos\theta}$$

$$\frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$\frac{\frac{2t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}=\frac{2t}{2}=t=\tan\frac{\theta}{2}$$

Answer $1-i\tan\left(\frac{1}{2}\theta\right)$