Algebra 3

This class deals with the roots of quadratic equations.

Roots of a Quadratic Equations

Note When we deal with the following the coefficient of x^2 must be 1, so if there is number in front of the x^2, we will divide the whole line by this number.

There follows three ways of doing the same thing

(i) A quadratic equation can be formed using

$$x^2 - (\text{sum of roots})x + \text{product of the roots} = 0$$

(ii) Sum of roots are equal to minus the coefficient of x

Product of roots is equal to the constant.

(iii) Given α and β are the roots of $ax^2 + bx + c = 0$

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Note We started with $ax^2 + bx + c = 0$ but since the coefficient of x^2 must be 1 we must divide across by a we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Type 1 To form a quadratic given the roots

Example 1 Form the quadratic equation with roots $3 \pm \sqrt{5}$

$\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$

Sum of roots $= \alpha + \beta = 3 + \sqrt{5} + 3 - \sqrt{5} = 6$

Product $= \alpha\beta = (3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 = 4$

Equation is $x^2 - 6x + 4 = 0$

Type 2 Evaluation questions

Here is a list of common questions that come up when we use α and β

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$	Learn.
$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ or $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$	Learn.
$\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$	Common denominator.
$\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$	Common denominator.
$(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$	Multiply out.
$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$	Factorise.
$\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$	Factorise.
$\begin{aligned}(\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= \alpha^2 + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 4\alpha\beta\end{aligned}$	Learn.

Example 2 If α and β are the roots of $4x^2 - 5x + 3 = 0$ without solving the equation find the values of

(i) $\alpha^2 + \beta^2$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

Before we can answer the question we must rearrange the original

$$4x^2 - 5x + 3 = 0$$

$$x^2 - \frac{5}{4}x + \frac{3}{4} = 0$$

$$\alpha + \beta = \frac{5}{4}$$

$$\alpha\beta = \frac{3}{4}$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{5}{4}\right)^2 - 2\left(\frac{3}{4}\right)$$

$$= \frac{25}{16} - \frac{6}{4} = \frac{1}{16}$$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

$$= \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{4} \times \frac{4}{3} = \frac{5}{3}$$

Type 3 Questions where the roots are related

Example 3 For what value of k will one of the roots of $x^2 + (3k+4)x + 3k + 4 = 0$ be three times the other root where $k \in R$.

Let one root be α then the other is 3α

sum of roots = - coefficient of x

$$3\alpha + \alpha = -(3k+4)$$
$$4\alpha = -(3k+4)$$
$$3k + 4 = -4\alpha$$

Product of roots = constant

$$\alpha(3\alpha) = 3k + 4$$
$$3k + 4 = 3\alpha^2$$

Note It is important that when you look at any question that you try to do it in the easiest way possible. This may mean that you stop for a minute to figure out the best way to go on.

Here we have two simultaneous equations, which we can solve by substituting from one into the other but since $3k+4 = 3k+4$ we get

$$3\alpha^2 = -4\alpha$$
$$3\alpha^2 + 4\alpha = 0$$
$$\alpha(3\alpha + 4) = 0$$
$$\alpha = 0 \quad \text{or} \quad \alpha = -\frac{4}{3}$$

Note If $\alpha = 0$ we cannot form a quadratic so we discard this answer.

Sub in $\alpha = -\frac{4}{3}$ into the equation $3k + 4 = -4\alpha$ to find $k = \frac{4}{9}$

Example 4 If one root is double the other in the equation $ax^2 + bx + c = 0$ show that $2b^2 = 9ac$

One root is α the other 2α

Sum of roots $\alpha + 2\alpha = -\frac{b}{a}$

$$3\alpha = -\frac{b}{a} \quad \text{- - - (i)}$$

Product of the root $\alpha(2\alpha) = \frac{c}{a}$

$$2\alpha^2 = \frac{c}{a} \quad \text{- - - (ii)}$$

Solve the simultaneous equations using substitution

From (i) $\alpha = -\frac{b}{3a}$ sub into (ii)

$$2\left(-\frac{b}{3a}\right)^2 = \frac{c}{a}$$

$$\frac{2b^2}{9a^2} = \frac{c}{a}$$ cross multiply

$$2ab^2 = 9a^2c$$ divide across by a

$$2b^2 = 9ac$$

Type 4 To form a quadratic which roots are related to a given quadratic

Example 5 If α and β are the roots of $2x^2 - 5x + 13 = 0$ construct a quadratic equation with roots $2\alpha - 1$ and $2\beta - 1$

$$\alpha + \beta = \frac{5}{2}$$
$$\alpha\beta = \frac{13}{2}$$

Sum of roots $2\alpha - 1 + 2\beta - 1$

$$= 2(\alpha + \beta) - 2$$
$$= 2\left(\frac{5}{2}\right) - 2 = 3$$

Product of roots $(2\alpha - 1)(2\beta - 1)$

$$= 4\alpha\beta - 2\alpha - 2\beta + 1$$
$$= 4\left(\frac{13}{2}\right) - 2\left(\frac{5}{2}\right) + 1 = 22$$

$$x^2 - (\text{sum of roots})x + \text{product of the roots} = 0$$

$$x^2 - 3x + 22 = 0$$

Example 6 If $\alpha + 3$ and $\beta + 3$ are the roots of $x^2 - 8x + 11 = 0$ form a quadratic with roots α^2 and β^2.

Note Usually the roots are α and β but here the roots are $\alpha + 3$ and $\beta + 3$.

Sum of roots $\alpha + 3 + \beta + 3 = 8$

$$\alpha + \beta = 2$$

Product of roots

$$(\alpha + 3)(\beta + 3) = 11$$
$$\alpha\beta + 3\alpha + 3\beta + 9 = 11$$
$$\alpha\beta + 3(\alpha + \beta) + 9 = 11$$
$$\alpha\beta + 3(2) + 9 = 11$$
$$\alpha\beta = -4$$

New quadratic

Sum of new roots $\alpha^2 + \beta^2$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$
$$= 2^2 - 2(-4) = 12$$

Product of roots $\alpha^2\beta^2$

$$= (\alpha\beta)^2$$
$$= (-4)^2 = 16$$

Required equation is $x^2 - 12x + 16 = 0$