Sequence and Series 6

This class goes through the natural number series arithmetic - geometric series and a couple of special cases.

The Natural Number Series

These are standard S_n series you are expected to be able to derive and to use.

(a) The sum of the first n natural numbers $\sum_{r=1}^{r=n} r = \frac{n}{2}(n+1)$

Note $\sum_{r=1}^{r=n} r = 1+2+3+4+---+n$

(b) Sum of the squares of the first n natural numbers $\sum_{r=1}^{r=n} r^2 = \frac{n}{6}(n+1)(2n+1)$

Note $\sum_{r=1}^{r=n} r^2 = 1^2+2^2+3^2+4^2+---+n^2$

(c) $\sum_{r=1}^{n} k = kn$

(d) $\sum_{r=1}^{n} kU_n = k\sum_{r=1}^{n} U_n$

Example 1 Find S_n of $2(5)+3(6)+4(7)+---$

Can you see that there are two different sequences being multiplied by each other. We need to find the general term of each sequence first.

For sequence 2, 3, 4, - - -

$$U_n = a+(n-1)d$$
$$= 2+1(n-1)$$
$$= n+1$$

For sequence 5, 6, 7, - - -

$$U_n = a+(n-1)d$$
$$= 5+1(n-1)$$
$$= n+4$$

$$U_n = (n+1)(n+4)$$

$$S_n = \sum_{n=1}^{n} (n+1)(n+4)$$

$$= \sum_{n=1}^{n} (n^2 + 5n + 4)$$

$$= \sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} 5n + \sum_{n=1}^{n} 4$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2} + 4n$$

$$= \frac{n}{6}((n+1)(2n+1) + 3(5)(n+1) + 6(4n))$$

$$= \frac{n}{6}(2n^2 + 3n + 1 + 15n + 15 + 24n)$$

$$= \frac{n}{6}(2n^2 + 42n + 16)$$

$$= \frac{n}{3}(n^2 + 21n + 8)$$

Example 2 Find $\sum_{n=1}^{n} (3n + 3^n + 3)$

$$\sum_{n=1}^{n} (3n + 3^n + 3) = \sum_{n=1}^{n} (3n) + \sum_{n=1}^{n} (3^n) + \sum_{n=1}^{n} (3)$$

$$\sum_{n=1}^{n} (3n) = 3\sum_{n=1}^{n} n$$

$$= \frac{3n(n+1)}{2}$$

$$= \frac{3n^2 + 3n}{2}$$

$$\sum_{n=1}^{n}(3^n) = 3^1 + 3^2 + 3^3 + - - - + 3^n$$

This is a geometric sequence where $a = 3$ and $r = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{3(3^n - 1)}{3 - 1}$$

$$= \frac{3}{2}(3^n - 1)$$

$$\sum_{n=1}^{n}(3) = 3n$$

$$\sum_{n=1}^{n}(3n + 3^n + 3) = \frac{3n^2 + 3n}{2} + \frac{3}{2}(3^n - 1) + 3n$$

$$= \frac{3n^2 + 9n + 3(3^n - 1)}{2}$$

Arithmetic - Geometric Series

This is a series where the coefficients are arithmetic and the variable is geometric.

Example 3 Find an expression for $\sum_{n \to 1}^{n} nx^n$ and hence find $\sum_{n \to 1}^{\infty} nx^n$ where $|x| < 1$.

$\sum_{n \to 1}^{n} nx^n = x + 2x^2 + 3x^3 + 4x^4 + - - - + nx^n$ and we need to find S_n

Let $S_n = x + 2x^2 + 3x^3 + 4x^4 + - - - + nx^n$ forget about the numbers in-front (the arithmetic part) and concentrate on the letters (the geometric part). Figure out what the common ratio is and multiply each term by this common ratio.

$$\begin{aligned} S_n &= x + 2x^2 + 3x^3 + 4x^4 + --- + nx^n \\ xS_n &= \quad x^2 + 2x^3 + 3x^4 + --- \quad + nx^{n+1} \\ \hline S_n - xS_n &= x \ + x^2 + \ x^3 + \ x^4 + --- + x^n - nx^{n+1} \end{aligned}$$

$x + x^2 + x^3 + x^4 + --- + x^n$ is a geometric sequence in which $a = x$ and the common ratio $r = x$. We can find S_n and then put back into our question.

$$S_n = \frac{x(1 - x^n)}{1 - x}$$

$$S_n - xS_n = x \ + x^2 + \ x^3 + \ x^4 + --- + x^n - nx^{n+1}$$

$$S_n(1 - x) = \frac{x(1 - x^n)}{1 - x} - nx^{n+1}$$

$$S_n = \frac{x(1 - x^n)}{(1 - x)^2} - \frac{nx^{n+1}}{1 - x}$$

$$\begin{aligned} \sum_{n=1}^{\infty} U_n &= \lim_{n \to \infty} S_n \\ &= \lim_{n \to \infty} \left(\frac{x(1 - x^n)}{(1 - x)^2} - \frac{nx^{n+1}}{1 - x} \right) \\ &= \frac{x}{(1 - x)^2} \end{aligned}$$

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Note From above we know that $\lim_{n \to \infty} x^n = 0$ given that $|x| < 1$

Special Case

Example 4 Find an expression for S_n of $5+55+555+5555+---$

Must first find an expression for the general term which is $U_n = 555555555$

$555555555 = 5 + (5\times10) + (5\times10^2) + (5\times10^3) + --$ This is a geometric series where $a = 5$ and $r = 10$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{5(10^n - 1)}{10 - 1}$$

$$= \frac{5}{9}(10^n - 1)$$

The general term of $5+55+555+5555+$ is $U_n = \frac{5}{9}(10^n - 1)$. We want to find S_n but remember

$$S_n = \sum_{n=1}^{n} U_n$$

$$S_n = \sum_{n=1}^{n} \frac{5}{9}(10^n - 1)$$

$$= \frac{5}{9}\sum_{n=1}^{n}(10^n - 1)$$

$$= \frac{5}{9}\left(\sum_{n=1}^{n} 10^n - \sum_{n=1}^{n} 1\right)$$

$$\sum_{n\to 1}^{n}\left(10^n\right)=10^1+10^2+10^3+---+10^n$$

This is a geometric sequence where $a=10$ and $r=10$

$$S_n=\frac{a\left(r^n-1\right)}{r-1}$$

$$=\frac{10\left(10^n-1\right)}{10-1}$$

$$=\frac{10}{9}\left(10^n-1\right)$$

$$\sum_{n\to 1}^{n}(1)=n$$

$$S_n=\sum_{n=1}^{n}\frac{5}{9}\left(10^n-1\right)$$

$$=\frac{5}{9}\left(\frac{10}{9}\left(10^n-1\right)-n\right)$$

Example 5 Find S_n of $1+(1+2)+(1+2+3)+---+(1+2+3+---+n)$

Must first find an expression for the general term which is
$U_n=1+2+3+---+n$

$1+2+3+---+n$ is arithmetic where $a=1$ and $d=1$

$$S_n=\frac{n}{2}\{2a+(n-1)d\}$$
$$=\frac{n}{2}(2+n-1)$$
$$=\frac{n}{2}(n+1)$$
$$=\frac{1}{2}\left(n^2+n\right)$$

The general term of $1+(1+2)+(1+2+3)+---+(1+2+3+---+n)$ is $U_n = \frac{1}{2}(n^2+n)$. We want to find S_n but remember

$$S_n = \sum_{n=1}^{n} U_n$$

$$S_n = \sum_{n=1}^{n} \frac{1}{2}(n^2+n)$$

$$= \frac{1}{2}\sum_{n=1}^{n}(n^2+n)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} n\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{12}(2n+1+3)$$

$$= \frac{n(n+1)(2n+4)}{12}$$

$$= \frac{n(n+1)(n+2)}{6}$$