Sequence and Series need to know 2

Sequence and Series 5

Recurrence Relation

This is where each term in a sequence depends on the terms before it i.e. U_{n+1} depends on U_n

Increasing and decreasing sequences

Given U_n is the nth term of a sequence then if

(i) $U_{n+1} > U_n$ for all n, the sequence is said to be increasing

$$\frac{U_{n+1}}{U_n} > 1 \text{ for all } n.$$

(ii) $U_{n+1} < U_n$ for all n, the sequence is said to be decreasing

$$\frac{U_{n+1}}{U_n} < 1 \text{ for all } n.$$

Infinite Series

We are dealing with a series of the form $U_1 + U_2 + U_3 + --- + U_\infty = \sum_{n=1}^{\infty} U_n$

$$S_\infty = \sum_{n=1}^{\infty} U_n = \lim_{n \to \infty} S_n$$

An arithmetic series can never have a limit and is therefore will always be divergent.

For a geometric series if $|r| < 1$ then $S_\infty = \dfrac{a}{1-r}$

For any other series to evaluate $S_\infty = \lim_{n \to \infty} S_n$

Step 1 Find a concise expression for S_n.

Step 2 Evaluate the limit of S_n as n approaches infinite.

To evaluate the limit use the rules from differentiation. One limit to know here is

$$\lim_{n\to\infty} r^n = 0 \text{ given that } |r| < 1$$

Telescoping Series

Find S_n and hence S_∞ of the series $\frac{1}{2(4)} + \frac{1}{3(5)} + \frac{1}{4(6)} + --- + \frac{1}{(n+1)(n+3)}$

$$U_n = \frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

Multiply across by the common denominator

$$1 = A(n+3) + B(n+1)$$
$$1 = An + 3A + Bn + B$$

$$A + B = 0$$
$$3A + B = 1$$

Solve the simultaneous equations to find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$

Note $A = -B$

Find $\sum_{n=1}^{n} \frac{1}{\sqrt{n+1} + \sqrt{n}}$

We have $U_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ and we are asked to find S_n.

This cannot be done the same way as above because of the square roots.

How can we change $U_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$. The answer is to multiply above and below by the conjugate of the bottom.

Sequence and Series 6

The Natural Number Series

These are standard S_n series you are expected to be able to derive and to use.

(a) The sum of the first n natural numbers $\sum_{r=1}^{r=n} r = \frac{n}{2}(n+1)$

Note $\sum_{r=1}^{r=n} r = 1+2+3+4+---+n$

(b) Sum of the squares of the first n natural numbers $\sum_{r=1}^{r=n} r^2 = \frac{n}{6}(n+1)(2n+1)$

Note $\sum_{r=1}^{r=n} r^2 = 1^2+2^2+3^2+4^2+---+n^2$

(c) $\sum_{r=1}^{n} k = kn$

(d) $\sum_{r=1}^{n} kU_n = k\sum_{r=1}^{n} U_n$

Arithmetic - Geometric Series

This is a series where the coefficients are arithmetic and the variable is geometric.

Find an expression for $\sum_{n\to1}^{n} nx^n$ and hence find $\sum_{n\to1}^{\infty} nx^n$ where $|x|<1$.

$\sum_{n\to1}^{n} nx^n = x+2x^2+3x^3+4x^4+---+nx^n$ and we need to find S_n

Let $S_n = x+2x^2+3x^3+4x^4+---+nx^n$ forget about the numbers in-front (the arithmetic part) and concentrate on the letters (the geometric part). Figure out what the common ratio is and multiply each term by this common ratio.

Note From above we know that $\lim_{n\to\infty} x^n = 0$ given that $|x|<1$.

Special Case

Find an expression for S_n of $5 + 55 + 555 + 5555 + ---$

Must first find an expression for the general term which is $U_n = 555555555$

The U_n for the whole series is a series itself, which we need to find S_n of.

Now we need to find the S_n of U_n.