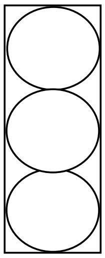
Area and Volume 5

This class contains shapes within shapes and double shapes.

Shapes within a shape

Example 1



3 spheres of radius 6cm are placed in the smallest possible cylinder. Find the percentage of empty space.

Empty space = volume of the cylinder – volume of the three spheres

Sphere $r = 6$

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(6)^3$$

$$= \frac{4}{3}\pi 216$$

$$= 288\pi$$

Volume of 3 spheres $= 3 \times 288\pi = 864\pi$

The main question here is what is the height of the cylinder. Each of the spheres has a radius of 6 and so a diameter of 12. The total height of the cylinder must be 36 since there is one sphere on top of the other and each sphere has a radius of 6 and thus a diameter of 12.

Find the volume of the cylinder $r = 6,\ h = 36$

$$V = \pi r^2 h$$

$$= \pi(6)^2(36)$$

$$= \pi(36)(36)$$

$$= 1296\pi$$

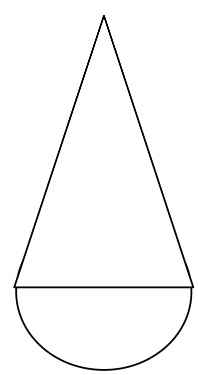
Empty space $1296\pi - 864\pi = 432\pi$

Percentage of empty space $= \dfrac{\text{space}}{\text{cylinder}} \times \dfrac{100}{1}$

$$= \frac{432\pi}{1296\pi} \times \frac{100}{1} = 33\%$$

Double Shapes

Example 2 A toy is made of a cone, which fits exactly on top of a hemisphere, as shown in the diagram. The radius length of the hemisphere is 6 cm and the total toy height is 21cm.



(i) Write down the height of the cone and hence find the volume of the cone in terms of π.

(ii) Find the volume of the hemisphere in terms of π.

(iii) Express the volume of the cone as a percentage of the volume of the total toy, to one decimal place.

With double shapes you must

State which two shapes you are dealing with.

Write down what you know about each shape, one length is missing (must be figured out).

Write down the formula and put figures in.

The total height of the toy is 21 cm and the height of the hemisphere is 6 cm so the cone must be 21 – 6 = 15 cm

Hemisphere $r = 6$

$$V = \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\pi(6)^3$$

$$= \frac{2}{3}\pi(216)$$

$$= 144\pi$$

Cone $r = 6$ and $h = 15$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(6)^2(15)$$

$$= \pi(36)(15)$$

$$= 1800\pi$$

Total $= 144\pi + 1800\pi = 1944\pi$

$$\text{Percentage cone } = \frac{\text{cone}}{\text{total}} \times \frac{100}{1}$$

$$= \frac{1800\pi}{1944\pi} \times \frac{100}{1}$$

$$= 92.59\%$$

$$= 92.6\%$$

Example 3 Wax in the shape of a cylinder with radius length 4 cm and height 36 cm is melted down. The resultant wax is formed into cone shaped candles. Each candle has a height 6 cm and a base radius length of 2 cm.

Calculate the number of candles that can be made, assuming that no wax is lost.
The candles are placed, base down and in rows of three, in the smallest possible rectangular box. Calculate, in cm^3, the volume of the box.
What percentage of the volume of the box is empty?

Cylinder $r = 4,\ h = 36$ find V.

$$V = \pi r^2 h$$
$$= \pi(4)^2(36)$$
$$= 576\pi$$

Cone $r = 2,\ h = 6$ find V

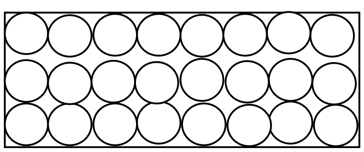
$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(2)^2(6)$$
$$= 24\pi$$

Number of candles $= \dfrac{\text{volume of cylinder}}{\text{volume of cone}} = \dfrac{576\pi}{24\pi} = 24$

There are 24 candles in rows of three so there must be 8 rows.
The box has three dimensions that we must figure out:

The height of the box is the same as the height of the candles = 6 cm.

To figure out the length and breath draw the box looking in from the top.



Each sphere has a radius of 2 so the thickness (diameter) of each cone is 4.

The length has 8 cones side by side so the total length must be $8 \times 4 = 32$.
The width has 3 cones side by side so the total width must be $3 \times 4 = 12$.

Box l=32, b = 12, h= 6 find V

$$V = l \times b \times h$$

$$= 32 \times 12 \times 6 = 2304$$

Empty space = volume of box – volume of 24 candles

$$= 2304 - 24(24\pi)$$
$$= 2304 - 24(24)(3.14)$$
$$= 2304 - 1808.64$$
$$= 495.36$$

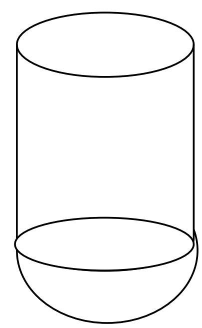
$$\text{Percentage empty} = \frac{\text{empty space}}{\text{volume of the box}} \times \frac{100}{1}$$

$$= \frac{495.36}{2304} \times \frac{100}{1} = 21.5\%$$

Example 4 A container is in the shape of a cylinder on top of a hemisphere as shown.
The cylinder has a radius of length 3 cm and the container has a total height of 15cm.

Calculate the volume of the container in terms of π.

If half the volume of the container is filled with liquid, calculate the height, h, of the liquid in the container.



Cylinder $r = 3,\ h = 12$ find V.

$$V = \pi r^2 h$$
$$= \pi(3)^2(12)$$
$$= 108\pi$$

Hemisphere $r = 3$

$$V = \frac{2}{3}\pi r^3$$
$$= \frac{2}{3}\pi(3)^3$$
$$= \frac{2}{3}\pi(27)$$
$$= 18\pi$$

Total volume = $108\pi + 18\pi = 126\pi$

Note The height of the cylinder was not given to us so we had to figure it out. The total height of the container is 15 but the height of the hemisphere at the bottom is 3. The height of the cylinder must be 15 – 3 =12

Half the volume of the container is filled with liquid, volume of water = 63π.

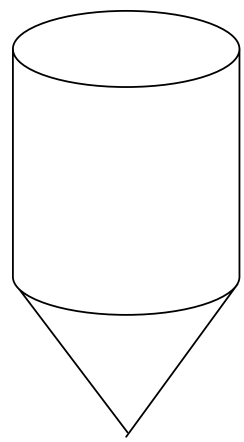
When the water is poured in it is the hemisphere at the bottom that gets filled up first.
The amount of water in the cylindrical part is $63\pi - 18\pi = 45\pi$.

Cylinder $r = 3, \; V = 45\pi$

$$V = \pi r^2 h$$
$$45\pi = \pi(3)^2(h)$$
$$9h = 45$$
$$h = 5$$

Height of water = 5 + 3 =8cm

Example 5 A grain- silo consists of a cylinder and an inverted cone, (as in diagram).
The height of the cylindrical part is 10 m and the radius is 2 m.
The slant height of the cone is 2.5 m.
Find the volume of the silo in terms of π.
When the volume of grain in the silo is 22π m^3, calculate the depth of grain measured from the apex (point) of the cone.



Cylinder $r = 2, \; h = 10$ find V.

$$V = \pi r^2 h$$
$$= \pi(2)^2(10)$$
$$= 40\pi$$

Cone $r = 2$, $l = 2.5$ find h

$$h^2 + r^2 = l^2 \text{ Use Pythagoras' theorem to solve for } h.$$
$$h^2 + 2^2 = (2.5)^2$$
$$h^2 + 4 = 6.25$$
$$h^2 = 2.25$$
$$h = 1.5$$

Cone $r = 2,\ h = 1.5$ find V

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(2)^2(1.5)$$
$$= 2\pi$$

Volume of container = $40\pi + 2\pi = 42\pi$

Volume of grain in the silo is 22π m^3.

The cone at the bottom gets filled up first. The rest of the grain goes into the cylindrical section. There must be $22\pi - 2\pi = 20\pi$ of grain in the cylindrical section.

We have a cylinder $r = 2,\ V = 20\pi$

$$V = \pi r^2 h$$
$$20\pi = \pi(2)^2(h)$$
$$4h = 20$$
$$h = 5$$

Total height of grain is $1.5 + 5 = 6.5$